Int. J. Solids Structures Vol. 29, No. 12, pp. 1519–1534, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

COMPOSITE CYLINDERS SUBJECTED TO HYGROTHERMAL AND MECHANICAL LOADS

LÁSZLÓ P. KOLLÁR,† JOCELYN M. PATTERSON and GEORGE S. SPRINGER
Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, U.S.A.

(Received 17 May 1991; in revised form 11 October 1991)

Abstract—The stress analysis of Kollár and Springer (1992) *Int. J. Solids Structures* **29**, 1499–1517, is applied to fiber-reinforced organic matrix composite cylinders subjected to hygrothermal and mechanical loads. The wall of the cylinder may be "thin" or "thick". An individual fiber must remain at the same radial distance from the axis; no other restrictions are placed either on fiber orientations or on stacking sequence. The applied loads may vary radially and circumferentially, but not axially. Solutions are presented which yield radially and circumferentially varying strains and stresses inside composite cylinders.

1. INTRODUCTION

In a previous paper (Kollár and Springer, 1992) we derived a set of equations which can be used to calculate the stresses and strains in anisotropic laminated composite cylinders and cylindrical segments. In the present paper we apply the analysis to composite cylinders subjected to hygrothermal and mechanical loads. The equations given by Kollár and Springer (referred to as Paper I) are not reproduced here. We only introduce those equations which are required for the solution of the problem and which are in addition to those given in the previous paper.

It is recalled that the solution is a three-dimensional elasticity solution of problems in which the loads, the stresses and the strains may vary radially and circumferentially, but not axially.

2. PROBLEM STATEMENT

We consider a cylinder made of n layers of unidirectional fiber reinforced composites (Fig. 1). There is no restriction on either the number of plies or the orientation (ply-angle) of the fibers in each ply. Hence the cylinder may be "thick" and the layup may be unsymmetric. However, the cylinder must be long, so that the length L is large compared to the thickness h and to the inner r^i and outer r^o radii ($h/L \ll 1$, $r^o/L \ll 1$, $r^i/L \ll 1$). This approximation implies that the edge effects are neglected.

The cylinder may be subjected to hygrothermal and mechanical loads which may vary in the radial r and circumferential θ directions, but must be independent of the axial coordinate x. Thus, the temperature ΔT and the moisture content Δc inside the composite may vary with r and θ but not with x. Here ΔT and Δc are the known temperature and moisture content relative to prescribed reference values T_r and c_r

$$\Delta T(\theta, r) = T - T_r \quad \Delta c(\theta, r) = c - c_r. \tag{1}$$

The temperature ΔT and the moisture Δc can be expressed in a Fourier series (Appendix A).

A mechanical load N_x may be imposed along the edge, as shown in Fig. 2. Axial, radial and circumferential surface loads (p_x^i, p_r^i, p_θ^i; p_x^o, p_r^o, p_θ^o) may also be applied on the inner and outer surfaces. All these loads may vary with θ but not with x. In addition, the cylinder may be subjected to a torque T and a bending moment M. The only restriction is that the

† On leave from the Technical University of Budapest, Department of Reinforced Concrete Structures, 1521 Hungary, Budapest.

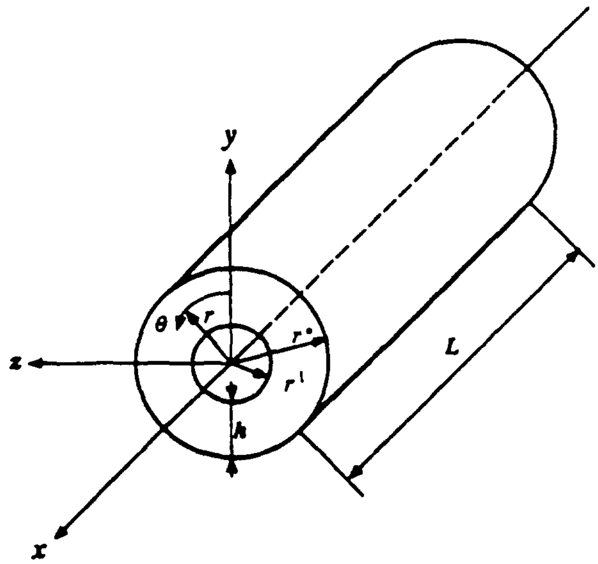

Fig. 1. Geometry of the closed cylinder.

mechanical loads must be in equilibrium, i.e. under their combined action the cylinder cannot undergo rigid body motion.

The inner and outer surfaces of the cylinder may be free or fixed, as shown in Fig. 3. On a fixed surface the displacements are zero; forces on such a surface cannot be specified. If neither the inner nor the outer surface is fixed, the applied forces must be in equilibrium and must satisfy the following equilibrium conditions:

force equilibrium in the x direction

$$\int_0^{2\pi} r^{o} p_x^{o}\, d\theta - \int_0^{2\pi} r^{i} p_x^{i}\, d\theta = 0; \tag{2}$$

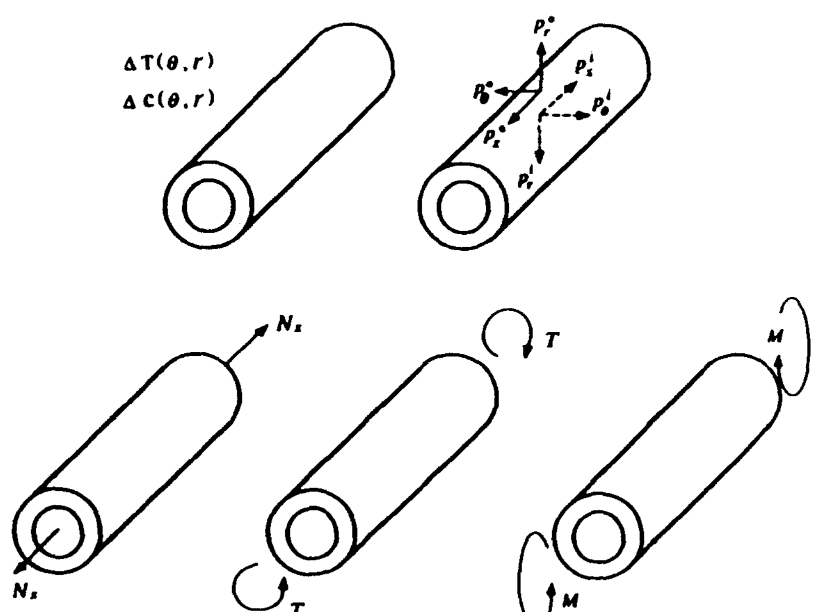

Fig. 2. Illustration of the loads on a closed cylinder.

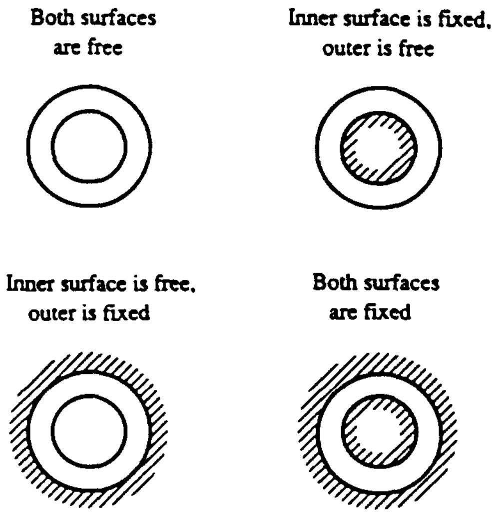

Fig. 3. The conditions on the inner and outer surfaces of a closed cylinder.

force equilibrium in the y direction

$$\int_0^{2\pi} r^{o}\{p_r^{o}\cos\theta - p_\theta^{o}\sin\theta\}\,d\theta - \int_0^{2\pi} r^{i}\{p_r^{i}\cos\theta - p_\theta^{i}\sin\theta\}\,d\theta = 0; \tag{3}$$

force equilibrium in the z direction

$$\int_0^{2\pi} r^{o}\{p_r^{o}\sin\theta - p_\theta^{o}\cos\theta\}\,d\theta - \int_0^{2\pi} r^{i}\{p_r^{i}\sin\theta - p_\theta^{i}\cos\theta\}\,d\theta = 0; \tag{4}$$

moment equilibrium about the x axis

$$\int_0^{2\pi} (r^{o})^2 p_\theta^{o}\,d\theta - \int_0^{2\pi} (r^{i})^2 p_\theta^{i}\,d\theta = 0. \tag{5}$$

The objective is to find the stresses and strains inside the composite, under the combined action of the temperature, moisture and mechanical loads.

3. DISPLACEMENTS

The axial, circumferential and radial displacements in the lth layer have the general form (eqn 1.7)†

$$\begin{aligned} u^l(x,\theta,r) &= u_o^l(x,\theta,r) + u_F^l(\theta,r) + u_B^l(x,\theta,r) \\ v^l(x,\theta,r) &= v_o^l(x,\theta,r) + v_F^l(\theta,r) + v_B^l(x,\theta,r) \\ w^l(x,\theta,r) &= w_o^l(x,\theta,r) + w_F^l(\theta,r) + w_B^l(x,\theta,r). \end{aligned} \tag{6}$$

The displacements u_o^l, v_o^l, w_o^l are given by eqn (1.9), u_F^l, v_F^l, w_F^l by eqn (1.15) and u_B^l, v_B^l, w_B^l by eqn (1.46). The superscript refers to the lth ply. These displacements contain a number of unknown constants in each layer (Table 1) which must be determined with the aid of the continuity conditions and the boundary conditions. The continuity conditions

† The numeral 1 preceding an equation number, a figure number or a table number refers to an equation, a figure or a table in Paper 1.

Table 1. The unknown constants in the displacements, continuity, periodicity, no rigid body motion and boundary conditions

	u_o^l, v_o^l, w_o^l	u_F^l, v_F^l, w_F^l $j \neq 1$	u_F^l, v_F^l, w_F^l $j = 1$	u_B^l, v_B^l, w_B^l
Unknowns (one layer)	A_1^l, A_2^l $u_a^l, u_b^l, u_c^l, u_d^l$ $v_a^l, v_b^l, v_c^l, v_d^l$	G_{jk}^l, $k = 1, 2, \ldots, 6$ G_{jk}^{l*}, $j = 1, 2, \ldots$, number of Fourier terms		κ^y, H_k^{ly} $k = 1, 2, \ldots, 6$ κ^z, H_k^{lz}
Number of unknowns	$10 * n$	$(2 * 6)\, n$ for each Fourier term		$2 + (2 * 6)\, n$
Continuity conditions	$10 * (n-1)$	$(2 * 6)\, (n-1)$ for each Fourier term		$(2 * 6)\, (n-1)$
Periodicity conditions	$u_b^1 = 0, \quad v_b^1 = 0$			
No rigid body motion	$u_d^1 = 0, \quad v_d^1 = 0$		$G_{j5}^1 = 0, \quad G_{j5}^{1*} = 0$	$H_5^{1y} = 0, \quad H_5^{1z} = 0$
Boundary conditions if there are no rigid body motions	6	$2 * 6$ for each Fourier term	$2 * 5$	$2 * 6$

are given in Tables 7–10 in Paper I, and are not reproduced here. The boundary conditions which must be applied in a specific problem are described subsequently. First, it is noted that for a closed cylinder certain periodicity conditions must also be satisfied. The periodicity conditions reflect the fact that each displacement must be a periodic function of θ, and must have the same value at θ and at $\theta + 2\pi$, i.e.

$$
\begin{aligned}
u^l(x, \theta, r) &= u^l(x, \theta + 2\pi, r), \\
v^l(x, \theta, r) &= v^l(x, \theta + 2\pi, r), \\
w^l(x, \theta, r) &= w^l(x, \theta + 2\pi, r).
\end{aligned} \tag{7}
$$

For u_o^l, v_o^l, w_o^l these conditions require that the following equalities be satisfied [eqn (I.9)]

$$u_b^l = 0, \quad v_b^l = 0. \tag{8}$$

When u_b^l and v_b^l are zero in any given ply, they must be zero in every ply by virtue of the continuity conditions (Table I.8). Thus, the periodicity conditions in eqn (8) become

$$u_b^1 = 0, \quad v_b^1 = 0. \tag{9}$$

To satisfy the requirements of eqn (7) we must also set θ_o equal to π in the expressions for u_F^l, v_F^l, w_F^l. Thus, u_F^l, v_F^l, w_F^l become [eqn (I.15)]

$$
\begin{aligned}
u_F^l(\theta, r) &= \sum_{j=1} \{u_j^l(r) \sin j\theta\} - \sum_{j=1} \{u_j^{l*}(r) \cos j\theta\} \\
v_F^l(\theta, r) &= \sum_{j=1} \{v_j^l(r) \sin j\theta\} - \sum_{j=1} \{v_j^{l*}(r) \cos j\theta\} \\
w_F^l(\theta, r) &= \sum_{j=1} \{w_j^l(r) \cos j\theta\} + \sum_{j=1} \{w_j^{l*}(r) \sin j\theta\}.
\end{aligned} \tag{10}
$$

Equations (9) are the periodicity conditions. The unknown constants in the expressions for the displacements (see Table 1) must be determined with the use of these periodicity conditions, together with the conditions of no rigid body motion, the continuity conditions, and the boundary conditions. The no rigid body motion and continuity conditions are specified in Paper I by eqns (56)–(58) and by Tables 7–10, respectively. The required boundary conditions are given in this paper.

4. GENERAL BOUNDARY CONDITIONS

In this section we present the general forms of the boundary conditions for a closed cylinder. We will then apply these boundary conditions to specific loading conditions.

4.1. *Axial load, torque and moment*

The axial load N_x, the torque T, and the M_y and M_z components of the bending moment in terms of the axial σ_x and shear stress $\tau_{x\theta}$ are (Figs 2 and 4)

$$N_x = \int_{r^i}^{r^o} \int_0^{2\pi} r\sigma_x(\theta, r)\,\mathrm{d}\theta\,\mathrm{d}r \tag{11}$$

$$T = \int_{r^i}^{r^o} \int_0^{2\pi} r^2\tau_{x\theta}(\theta, r)\,\mathrm{d}\theta\,\mathrm{d}r \tag{12}$$

$$M_y = \int_{r^i}^{r^o} \int_0^{2\pi} r^2\sigma_x(\theta, r)\cos\theta\,\mathrm{d}\theta\,\mathrm{d}r \tag{13}$$

$$M_z = \int_{r^i}^{r^o} \int_0^{2\pi} r^2\sigma_x(\theta, r)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r \tag{14}$$

where M_y and M_z are the components of the applied bending moments in the x–y and x–z planes, respectively (Fig. 4).

4.2. *Surface loads*

Axial p_x, radial p_r, and circumferential loads p_θ can be applied on the inner and outer surfaces. As stated before each of these loads may vary with θ. On the inner and outer surfaces the applied surface loads are related to the stresses by the expressions

$$\begin{aligned} p_r^i(\theta) &= \sigma_r^i(\theta, r) \\ p_\theta^i(\theta) &= \tau_{r\theta}^i(\theta, r), \quad r = r^i \\ p_x^i(\theta) &= \tau_{rx}^i(\theta, r) \end{aligned} \tag{15}$$

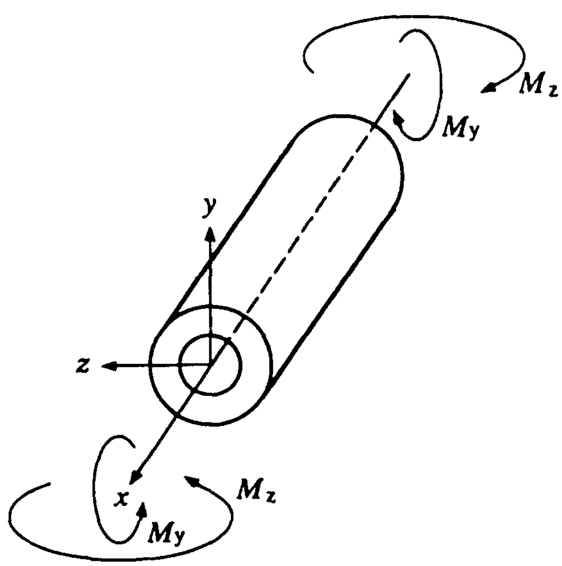

Fig. 4. Components of the bending moment M.

$$p_r^o(\theta) = \sigma_r^n(\theta, r)$$
$$p_\theta^o(\theta) = \tau_{r\theta}^n(\theta, r), \quad r = r^o.$$
$$p_x^o(\theta) = \tau_{rx}^n(\theta, r) \tag{16}$$

The superscripts i and o refer to the inner and outer surfaces, and 1 and n to the innermost ($l = 1$) and outermost ($l = n$) plies.

The surface loads may be expressed in terms of the Fourier series

$$p_r^{i,o}(\theta) = \sum_{j=0} \{\hat{p}_{rj}^{i,o} \cos j\theta\} + \sum_{j=0} \{\hat{p}_{rj}^{i,o*} \sin j\theta\}$$
$$p_\theta^{i,o}(\theta) = \sum_{j=0} \{\hat{p}_{\theta j}^{i,o} \sin j\theta\} + \sum_{j=0} \{\hat{p}_{\theta j}^{i,o*} \cos j\theta\}$$
$$p_x^{i,o}(\theta) = \sum_{j=0} \{\hat{p}_{xj}^{i,o} \sin j\theta\} + \sum_{j=0} \{\hat{p}_{xj}^{i,o*} \cos j\theta\}. \tag{17}$$

The manner in which the constants $\hat{p}$ and $\hat{p}^*$ are determined is discussed in Appendix A [eqns (A5–A7)]. Equations (17) apply at both the inner (superscript i) and outer surfaces (superscript o). Substituting these surface forces into eqns (2–5) the equilibrium conditions can be expressed in terms of $\hat{p}$ and $\hat{p}^*$. The results are summarized in Appendix B.

The stresses may also be written in series form, as shown in Table 1.5. The following four stress components, taken from Table 1.5, are of interest here

$$\sigma_x(\theta, r) = \sigma_{x0} + \sum_{j=1} \{\hat{\sigma}_{xj} \cos j\theta\} + \sum_{j=1} \{\hat{\sigma}_{xj}^* \sin j\theta\} + \hat{\sigma}_{xB}^c \cos\theta + \hat{\sigma}_{xB}^s \sin\theta$$
$$\sigma_r(\theta, r) = \sigma_{r0} + \sum_{j=1} \{\hat{\sigma}_{rj} \cos j\theta\} + \sum_{j=1} \{\hat{\sigma}_{rj}^* \sin j\theta\} + \hat{\sigma}_{rB}^c \cos\theta + \hat{\sigma}_{rB}^s \sin\theta$$
$$\tau_{r\theta}(\theta, r) = \tau_{r\theta 0} + \sum_{j=1} \{\hat{\tau}_{r\theta j} \sin j\theta\} + \sum_{j=1} \{\hat{\tau}_{r\theta j}^* \cos j\theta\} + \hat{\tau}_{r\theta B}^c \sin\theta + \hat{\tau}_{r\theta B}^s \cos\theta$$
$$\tau_{rx}(\theta, r) = \tau_{rx0} + \sum_{j=1} \{\hat{\tau}_{rxj} \sin j\theta\} + \sum_{j=1} \{\hat{\tau}_{rxj}^* \cos j\theta\} + \hat{\tau}_{rxB}^c \sin\theta + \hat{\tau}_{rxB}^s \cos\theta. \tag{18}$$

Comparisons of eqns (15)–(18) yield the boundary conditions for the stresses. The results for $\hat{p}$ and $\hat{p}^*$ are summarized in Table 2.

Because of the equilibrium conditions [eqns (2–5) and Appendix B] the surface loads (and correspondingly the surface stresses) are not all independent when both surfaces are free. From the equilibrium conditions, given in Appendix B, it is seen that for $j = 0$ only four (instead of the six), and for $j = 1$ only 10 (of the 12) boundary forces are independent. Thus, two of the boundary conditions for $j = 0$, and $j = 1$ cannot be used. These are indicated by the brackets in Table 2.

4.3. *Surface displacements*

The inner or outer surfaces may be fixed (Fig. 3). When the inner surface is fixed, the displacements on the inner surface ($r = r^i$) are

$$u^1(x, \theta, r) = 0, \quad v^1(x, \theta, r) = 0, \quad w^1(x, \theta, r) = 0. \tag{25}$$

When the outer surface is fixed, the displacements on the outer surface ($r = r^o$) are

$$u^n(x, \theta, r) = 0, \quad v^n(x, \theta, r) = 0, \quad w^n(x, \theta, r) = 0. \tag{26}$$

The application of the above boundary conditions to specific loading conditions are described below.

Table 2. The stress boundary conditions

For $j = 0$

$$\left.\begin{aligned} \dot{p}^{i}_{r0} &= \sigma^{i}_{r0} \\ \dot{p}^{i*}_{\theta 0} &= \tau^{i}_{r\theta 0} \\ \dot{p}^{i*}_{x0} &= \tau^{i}_{rx0} \end{aligned}\right\} \quad r = r^{i} \tag{19}$$

$$\left.\begin{aligned} \dot{p}^{o}_{r0} &= \sigma^{n}_{r0} \\ (\dot{p}^{o*}_{\theta 0} &= \tau^{n}_{r\theta 0}) \\ (\dot{p}^{o*}_{x0} &= \tau^{n}_{rx0}) \end{aligned}\right\} \quad r = r^{o}. \tag{20}$$

For $j = 1$

$$\left.\begin{aligned} \dot{p}^{i}_{r1} &= \dot{\sigma}^{i}_{r1} + \dot{\sigma}^{is}_{rB}, & \dot{p}^{i*}_{r1} &= \dot{\sigma}^{i*}_{r1} + \dot{\sigma}^{ic}_{rB} \\ \dot{p}^{i}_{\theta 1} &= \dot{\tau}^{i}_{r\theta 1} + \dot{\tau}^{is}_{r\theta B}, & \dot{p}^{i*}_{\theta 1} &= \dot{\tau}^{i*}_{r\theta 1} + \dot{\tau}^{ic}_{r\theta B} \\ \dot{p}^{i}_{x1} &= \dot{\tau}^{i}_{rx1} + \dot{\tau}^{is}_{rxB}, & \dot{p}^{i*}_{x1} &= \dot{\tau}^{i*}_{rx1} + \dot{\tau}^{ic}_{rxB} \end{aligned}\right\} \quad r = r^{i} \tag{21}$$

$$\left.\begin{aligned} \dot{p}^{o}_{r1} &= \dot{\sigma}^{n}_{r1} + \dot{\sigma}^{ns}_{rB}, & \dot{p}^{o*}_{r1} &= \dot{\sigma}^{n*}_{r1} + \dot{\sigma}^{nc}_{rB} \\ (\dot{p}^{o}_{\theta 1} &= \dot{\tau}^{n}_{r\theta 1} + \dot{\tau}^{ns}_{r\theta B}), & (\dot{p}^{o*}_{\theta 1} &= \dot{\tau}^{n*}_{r\theta 1} + \dot{\tau}^{nc}_{r\theta B}) \\ \dot{p}^{o}_{x1} &= \dot{\tau}^{n}_{rx1} + \dot{\tau}^{ns}_{rxB}, & \dot{p}^{o*}_{x1} &= \dot{\tau}^{n*}_{rx1} + \dot{\tau}^{nc}_{rxB} \end{aligned}\right\} \quad r = r^{o}. \tag{22}$$

For $j \geq 2$

$$\left.\begin{aligned} \dot{p}^{i}_{rj} &= \dot{\sigma}^{i}_{rj}, & \dot{p}^{i*}_{rj} &= \dot{\sigma}^{i*}_{rj} \\ \dot{p}^{i}_{\theta j} &= \dot{\tau}^{i}_{r\theta j}, & \dot{p}^{i*}_{\theta j} &= \dot{\tau}^{i*}_{r\theta j} \\ \dot{p}^{i}_{xj} &= \dot{\tau}^{i}_{rxj}, & \dot{p}^{i*}_{xj} &= \dot{\tau}^{i*}_{rxj} \end{aligned}\right\} \quad r = r^{i} \tag{23}$$

$$\left.\begin{aligned} \dot{p}^{o}_{rj} &= \dot{\sigma}^{1}_{rj}, & \dot{p}^{o*}_{rj} &= \dot{\sigma}^{n*}_{rj} \\ \dot{p}^{o}_{\theta j} &= \dot{\tau}^{1}_{r\theta j}, & \dot{p}^{o*}_{\theta j} &= \dot{\tau}^{n*}_{r\theta j} \\ \dot{p}^{o}_{xj} &= \dot{\tau}^{1}_{rxj}, & \dot{p}^{o*}_{xj} &= \dot{\tau}^{n*}_{rxj} \end{aligned}\right\} \quad r = r^{o}. \tag{24}$$

5. AXIAL LOAD AND TORQUE

When only an axial force N_x and/or a torque T are applied the strains are independent of θ and are a function of r only. In this case the displacements are [eqn (19)]

$$u^{l} = u^{l}_{o}, \quad v^{l} = v^{l}_{o}, \quad w^{l} = w^{l}_{o}. \tag{27}$$

The required six boundary conditions are as follows. For the axial load and the torque we have [eqns (11–12)]

$$2\pi \int_{r^{i}}^{r^{o}} r\sigma_{xo}(r)\,\mathrm{d}r = N_x$$

$$2\pi \int_{r^{i}}^{r^{o}} r^{2}\tau_{x\theta o}(r)\,\mathrm{d}r = T. \tag{28}$$

The subscript o indicates that the stresses correspond to the u^{l}_{o}, v^{l}_{o}, w^{l}_{o} displacements (Table 1.5). For $u^{l}_{o}, v^{l}_{o}, w^{l}_{o}$ the bending moments M_y [eqn (13)] and M_z [eqn (14)] are zero ($M_y = M_z = 0$).

In the absence of surface loads, the boundary conditions on the inner and outer surfaces are [eqns (19) and (20)]

$$\left.\begin{aligned} \sigma^{1}_{ro} &= 0 \\ \tau^{1}_{r\theta o} &= 0, \quad r = r^{i} \\ \tau^{1}_{rxo} &= 0 \end{aligned}\right. \tag{29}$$

$$\sigma^{n}_{ro} = 0, \quad r = r^{o}. \tag{30}$$

6. BENDING

The cylinder is subjected to a pure bending moment M which has two components M_x and M_z (Fig. 4). By our definition [eqn (1.7)] the displacements in each layer are

$$u^l = u_B^l, \quad v^l = v_B^l, \quad w^l = w_B^l. \tag{31}$$

The required 12 boundary conditions are as follows. The first two boundary conditions are obtained by substituting the appropriate expressions for the stresses [eqn (18)] into eqns (13) and (14), and by integrating with respect to θ. The results are

$$\pi\int_{r'}^{r''} r^2\hat{\sigma}_{xB}^{y}(r)\,dr = M_y$$

$$\pi\int_{r'}^{r''} r^2\hat{\sigma}_{xB}^{z}(r)\,dr = M_z. \tag{32}$$

For u_B^l, v_B^l, w_B^l the axial force N_x [eqn (11)] and the torque T [eqn (12)] are zero ($N_x = 0$, $T_x = 0$).

The displacements u_F^l, v_F^l, w_F^l are zero. The stresses ($\hat{\sigma}_{rl}, \ldots, \hat{\tau}_{rx}^{*}$) corresponding to these displacements are then also zero. Thus, in the absence of surface loads, the boundary conditions on the inner and outer surfaces are

inner surface ($r = r'$) [eqn (21)]

$$\hat{\sigma}_{rB}^{1y} = 0, \quad \hat{\sigma}_{rB}^{1z} = 0$$

$$\hat{\tau}_{r\theta B}^{1y} = 0, \quad \hat{\tau}_{r\theta B}^{1z} = 0$$

$$\hat{\tau}_{rxB}^{1y} = 0, \quad \hat{\tau}_{rxB}^{1z} = 0 \tag{33}$$

outer surface ($r = r''$) [eqn (22)]

$$\hat{\sigma}_{rB}^{ny} = 0, \quad \hat{\sigma}_{rB}^{nz} = 0$$

$$\hat{\tau}_{rxB}^{ny} = 0, \quad \hat{\tau}_{rxB}^{nz} = 0. \tag{34}$$

7. SURFACE LOADS

A cylinder is subjected to normal and tangential surface loads at the inner and outer surfaces (Fig. 2b). Under the action of these loads the displacements are [eqn (6)]

$$u^l = u_o^l + u_F^l + u_B^l, \quad v^l = v_B^l + v_F^l + v_B^l, \quad w^l = w_B^l + w_F^l + w_B^l. \tag{35}$$

We obtain the solution to this problem in two steps. In the first step we calculate the displacements under the restriction that the axis of cylinder remains straight. This results in the M_y^* and M_z^* bending moments shown in Fig. 5. In the second step we calculate the displacements in the cylinder when subjected to bending moments which have the same

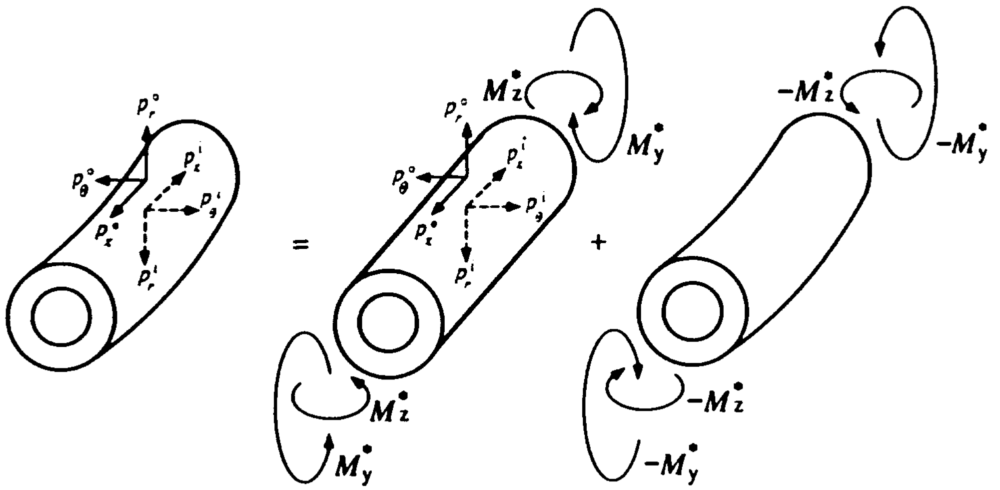

Fig. 5. Solution steps for a cylinder subjected to surface loads.

absolute values as M_y^* and M_z^* but are of the opposite sense. The final displacements are the sum of the displacements given by solution steps one and two.

7.1. *Axis remains straight*

When the axis remains straight u_B^l, v_B^l, w_B^l are zero, and the displacements are

$$u^l = u_0^l + u_F^l, \quad v^l = v_0^l + v_F^l, \quad w^l = w_0^l + w_F^l \tag{36}$$

where u_0^l, v_0^l, w_0^l are given by eqns (1.9) and u_F^l, v_F^l, w_F^l by eqn (10).

Boundary conditions for u_0^l, v_0^l, w_0^l. In the absence of an axial load N_x and a torque T eqns (11) and (12) yield

$$2\pi \int_{r^i}^{r^o} r\sigma_{x0}(r)\,\mathrm{d}r = 0$$

$$2\pi \int_{r^i}^{r^o} r^2\tau_{x\theta 0}(r)\,\mathrm{d}r = 0. \tag{37}$$

As was noted before, for u_0^l, v_0^l, w_0^l the bending moments M_y [eqn (13)] and M_z [eqn (14)] are zero ($M_y = M_z = 0$). The boundary conditions for the surface loads at the inner and outer surfaces are [eqns (19) and (20)]

$$\left.\begin{aligned} \sigma_{r0}^{i} &= \hat{p}_{r0}^{i} \\ \tau_{r\theta 0}^{i} &= \hat{p}_{\theta 0}^{i*} \quad r = r^i \\ \tau_{rx0}^{i} &= \hat{p}_{x0}^{i*} \end{aligned}\right. \tag{38}$$

$$\sigma_{r0}^{n} = \hat{p}_{r0}^{o} \quad r = r^o. \tag{39}$$

Boundary conditions for u_F^l, v_F^l, w_F^l ($j = 1$). Next we examine the boundary conditions required to determine the constants in the $j = 1$ terms of the series u_F^l, v_F^l, w_F^l. Referring to eqns (21) and (22), the boundary conditions are as follows:

Since the axis remains straight u_B^j, v_B^j, w_B^j are zero. The corresponding stresses $(\hat{\sigma}_{rB}^j, \ldots, \hat{\tau}_{rxB}^j)$ are also zero. Thus, from eqns (21) and (22) the boundary conditions at the inner and outer surfaces are:

inner surface ($r = r^i$) [eqn (21)]

$$\begin{aligned} \hat{\sigma}_{r1}^{1} &= \hat{p}_{r1}^{i}, \quad \hat{\sigma}_{r1}^{1*} = \hat{p}_{r1}^{i*} \\ \hat{\tau}_{r\theta 1}^{1} &= \hat{p}_{\theta 1}^{i}, \quad \hat{\tau}_{r\theta 1}^{1*} = \hat{p}_{\theta 1}^{i*} \\ \hat{\tau}_{rx1}^{1} &= \hat{p}_{x1}^{i}, \quad \hat{\tau}_{rx1}^{1*} = \hat{p}_{x1}^{i*} \end{aligned} \tag{40}$$

outer surface ($r = r^o$) [eqn (22)]

$$\begin{aligned} \hat{\sigma}_{r1}^{n} &= \hat{p}_{r1}^{o}, \quad \hat{\sigma}_{r1}^{n*} = \hat{p}_{r1}^{o*} \\ \hat{\tau}_{rx1}^{n} &= \hat{p}_{x1}^{o}, \quad \hat{\tau}_{rx1}^{n*} = \hat{p}_{x1}^{o*}. \end{aligned} \tag{41}$$

Once all the constants in the displacements are known the stresses can be calculated. The bending moments M_y^* and M_z^* are then evaluated by

$$\begin{aligned} M_y^* &= \pi \int_{r^i}^{r^o} r^2 \hat{\sigma}_{x1}(r)\,\mathrm{d}r \\ M_z^* &= \pi \int_{r^i}^{r^o} r^2 \hat{\sigma}_{x1}^*(r)\,\mathrm{d}r. \end{aligned} \tag{42}$$

For u_F^j, v_F^j, w_F^j ($j = 1$) the axial force N_x [eqn (11)] and the torque T [eqn (12)] are zero ($N_x = 0$, $T = 0$).

Boundary conditions for u_F^j, v_F^j, w_F^j ($j \geqslant 2$). Next we examine the boundary conditions required to determine the constants in the $j \geqslant 2$ terms of the series u_F^j, v_F^j, w_F^j. The boundary conditions at the inner and outer surfaces are:

inner surface ($r = r^i$) [eqn (23)]

$$\begin{aligned} \hat{\sigma}_{rj}^{1} &= \hat{p}_{rj}^{i}, \quad \hat{\sigma}_{rj}^{1*} = \hat{p}_{rj}^{i*} \\ \hat{\tau}_{r\theta j}^{1} &= \hat{p}_{\theta j}^{i}, \quad \hat{\tau}_{r\theta j}^{1*} = \hat{p}_{\theta j}^{i*} \\ \hat{\tau}_{rxj}^{1} &= \hat{p}_{rxj}^{i}, \quad \hat{\tau}_{rxj}^{1*} = \hat{p}_{xj}^{i*} \end{aligned} \tag{43}$$

outer surface ($r = r^o$) [eqn (24)]

$$\begin{aligned} \hat{\sigma}_{rj}^{n} &= \hat{p}_{rj}^{o}, \quad \hat{\sigma}_{rj}^{n*} = \hat{p}_{rj}^{o*} \\ \hat{\tau}_{r\theta j}^{n} &= \hat{p}_{\theta j}^{o}, \quad \hat{\tau}_{r\theta j}^{n*} = \hat{p}_{\theta j}^{o*} \\ \hat{\tau}_{rxj}^{n} &= \hat{p}_{xj}^{o}, \quad \hat{\tau}_{rxj}^{n*} = \hat{p}_{xj}^{o*}. \end{aligned} \tag{44}$$

The axial force N_x [eqn (11)], the torque T [eqn (12)], and the bending moments M_y [eqn (13)] and M_z [eqn (14)] are zero when $j \geqslant 2$ ($N_x = 0$, $T = 0$, $M_z = 0$, $M_y = 0$).

7.2. *Curved axis*

To complete the problem we impose the bending moments $-M_y^*$ and $-M_z^*$ [eqn (42)] on the cylinder, as shown in Fig. 5. The calculations then proceed along the lines described above for pure bending [eqns (31)–(34)].

7.3. *Displacements under surface loads*

The complete displacement field is obtained by summing all the displacements for the case when the axis remains straight and for the case when the axis is curved.

8. INNER AND OR OUTER SURFACES FIXED

Finally we consider problems in which either the inner surface, the outer surface, or both of these surfaces are fixed. When either of these surfaces is fixed the axis must remain straight and the displacements in each ply are [eqn (6)]

$$u^l = u_o^l + u_F^l, \quad v^l = v_o^l + v_F^l, \quad w^l = w_o^l + w_F^l. \tag{45}$$

Below we describe the boundary conditions for a cylinder with the inner surface fixed and surface loads applied on the outer surface. In this case the displacements of the inner surface are zero ($r = r^i$)

$$u^1(x,\theta,r) = 0, \quad v^1(x,\theta,r) = 0, \quad w^1(x,\theta,r) = 0. \tag{46}$$

The surface loads on the outer surfaces are ($r = r^o$) [eqn (16)]

$$\begin{aligned} p_r^o(\theta) &= \sigma_r^n(\theta,r) \\ p_\theta^o(\theta) &= \tau_{r\theta}^n(\theta,r) \\ p_x^o(\theta) &= \tau_{rx}^n(\theta,r). \end{aligned} \tag{47}$$

From eqns (46) and (6) we have ($r = r^i$)

$$u_o^1(x,\theta,r) = 0, \quad v_o^1(x,\theta,r) = 0, \quad w_o^1(r) = 0 \tag{48}$$

$$u_F^1(\theta,r) = 0, \quad v_F^1(\theta,r) = 0, \quad w_F^1(\theta,r) = 0. \tag{49}$$

Boundary conditions for u_o^l, v_o^l, w_o^l ($j = 0$)

The displacements u_o^l, v_o^l, w_o^l are [eqn (1.9)]

$$\begin{aligned} u_o^l &= u_a^l x + u_b^l\theta + u_c^l(r) \\ v_o^l &= v_a^l xr + v_b^l\theta r + v_c^l(r) \\ w_o^l &= w_o^l(r) \end{aligned} \tag{50}$$

where u_a, u_b, v_a, v_b are constants, while $u_c^l(r)$, $v_c^l(r)$, $w_o^l(r)$ are given by eqns (1.11)–(1.13). Due to the periodicity conditions [eqn (8)] u_b^l and v_b^l are zero. From eqns (48) and (50) we obtain the following boundary conditions for the displacements at the inner surface ($r = r^i$)

$$\begin{aligned} u_a^1 &= 0, \quad u_c^1(r) = 0 \\ v_a^1 &= 0, \quad v_c^1(r) = 0 \\ w_o^1(r) &= 0. \end{aligned} \tag{51}$$

The boundary conditions at the outer surface ($r = r^o$) are [see eqn (20)]

$$\begin{aligned} \sigma_{ro}^n &= \hat{p}_{ro}^o \\ \tau_{r\theta o}^n &= \hat{p}_{\theta o}^{o*} \\ \tau_{rxo}^n &= \hat{p}_{xo}^{o*}. \end{aligned} \tag{52}$$

There are eight boundary conditions [eqns (51) and (52)] instead of the six specified in Table 1. The reason for this is that the restriction of no rigid body motion is incorporated in the displacement boundary conditions [eqn (51)]. Hence, the condition of no rigid body motions ($u_J^1 = v_J^1 = 0$ in Table 1) is now not used separately.

Boundary conditions for u_F^l, v_F^l, w_F^l ($j \geq 1$)

Next we examine the boundary conditions for the $j \geq 1$ terms of the series representing u_F^l, v_F^l, w_F^l. By comparing eqns (10) and (49) we obtain the following six boundary conditions for the displacements at the inner surface ($r = r^i$)

$$u_j^1(r) = 0, \quad u_j^{1*}(r) = 0$$
$$v_j^1(r) = 0, \quad v_j^{1*}(r) = 0$$
$$w_j^1(r) = 0, \quad w_j^{1*}(r) = 0. \tag{53}$$

The boundary conditions at the outer surface ($r = r^o$) are [see eqn (24)]

$$\hat{\sigma}_{rj}^n = \hat{p}_{rj}^o, \quad \hat{\sigma}_{rj}^{n*} = \hat{p}_{rj}^{o*}$$
$$\hat{\tau}_{r\theta j}^n = \hat{p}_{\theta j}^o, \quad \hat{\tau}_{r\theta j}^{n*} = \hat{p}_{\theta j}^{o*}$$
$$\hat{\tau}_{rxj}^n = \hat{p}_{xj}^o, \quad \hat{\tau}_{rxj}^{n*} = \hat{p}_{xj}^{o*}. \tag{54}$$

There are 12 boundary conditions [eqns (53) and (54)] for every jth term of the series. Table 1 indicates the need for only 10 boundary conditions when $j = 1$. The additional two boundary conditions are introduced because the condition of no rigid body motion ($G_{j\varsigma}^1 = G_{j\varsigma}^{1*} = 0$) is not used separately.

The above boundary conditions apply when the inner surface is fixed. The boundary conditions can be established in an identical manner when either the outer surface is fixed, or if both the inner and outer surfaces are fixed.

Table 3. The input parameters required for the CYLINDER computer code

Geometry
—inner radius, r^i
—thickness of one ply, h_0
—number of ply groups, n
—number of plies in each ply group, m_i
—direction (angle) of fibers in each ply group, ϕ_i
Material (on-axis) properties
—Longitudinal Young's modulus, E_1
—Transverse in-plane Young's modulus, E_2
—Transverse out-of-plane Young's modulus, E_3
—Shear moduli, G_{23}, G_{13}, G_{12}
—Poisson's ratios, v_{23}, v_{13}, v_{12}
—Longitudinal thermal expansion coefficient, α_1
—Transverse in-plane thermal expansion coefficient, α_2
—Transverse out-of-plane thermal expansion coefficient, α_3
Loads
—Tensile force, N_x
—Torque, T
—Bending moment in the x–y plane, M_y
—Bending moment in the x–z plane, M_z
—Surface loads on the inner surface, p_r^i, p_θ^i, p_x^i
—Surface loads on the outer surface, p_r^o, p_θ^o, p_x^o
—Temperature, ΔT
Constraint
—Displacements along the inner surface
—Displacements along the outer surface.

Table 4. The output parameters for the CYLINDER computer code

—Strains in each ply, ε_x, ε_θ, ε_r, $\gamma_{\theta r}$, γ_{xr}, $\gamma_{\theta x}$
—Stresses in each ply, σ_x, σ_θ, σ_r, $\tau_{\theta r}$, τ_{xr}, $\tau_{\theta x}$
—Displacements in each ply, u, v, w.

9. METHOD OF SOLUTION

The equations presented in Paper I together with those given in the present paper form a complete set for calculating the displacements, stresses and strains under specified boundary conditions. Solutions of these equations require extensive analytical and numerical calculations.

The analytical calculations were performed with a symbolic manipulator, "Mathematica" (Wolfram, 1988). This resulted in a set of simultaneous algebraic equations for the unknown constants in the displacements. The constants were determined from these equations. A computer code (designated as CYLINDER) was written for performing the calculations. The input parameters required for and the output parameters provided by the code are given in Tables 3 and 4.

Typical computational time on a Macintosh II personal computer for a hundred ply laminate is approximately eight minutes.

10. SAMPLE PROBLEM

Solutions were obtained for a sample problem to illustrate the output provided by the model and the CYLINDER code. A problem similar to that given by Hyer *et al.* (1986) was analysed. The cylinder consists of four plies. The fibers in the first (innermost) and third plies are at an angle ϕ with the axis of the cylinder (Fig. 6). The fibers in the second and fourth (outermost) plies are orthogonal to those in the other two plies, i.e. the fiber orientation in the second and fourth plies is $\phi + 90°$.

The analysis of Hyer *et al.* (1986) is applicable only to cross-ply cylinders, and hence the results given by Hyer *et al.* are only for $\phi = 0°$ and $\phi = 90°$. Our analysis is applicable to general anisotropy, and we calculated results for cylinders with ϕ varying from 0° to 90°. The material properties used in the calculation were the same as those used by Hyer *et al.* and are given in Table 5.

For the two special cases of $\phi = 0°$ and $\phi = 90°$ the displacements and the stresses calculated by the present model were the same as those obtained by Hyer *et al.*

We have also calculated the angular and axial deformations of the cylinder as a function of ϕ (Fig. 6). As expected, these deformations depend on the ply orientation ϕ. In the present problem the effect of the ply orientation ϕ on the axial deformation is small because

Table 5. Data for the sample problem

Geometry
—inner radius, $r^i = 6.35$ mm
—thickness of one ply, $h_0 = 0.127$ mm
—number of plies, $n = 4$
—ply orientatation, $[\phi/90+\phi/\phi/90+\phi]$, $(0° \leqslant \phi \leqslant 90°)$

Material (on-axis) properties
—Young's moduli, $E_1 = 146.8$ GPa
$E_2 = 9.929$ GPa
$E_3 = 9.101$ GPa
—Shear moduli, $G_{23} = 3.05$ GPa
$G_{13} = G_{12} = 7.17$ GPa
—Poisson's ratios, $v_{23} = 0.49$
$v_{13} = v_{12} = 0.3$
—Thermal expansion coefficients, $\alpha_1 = -0.00774 \cdot 10^{-6}$ °C^{-1}
$\alpha_2 = \alpha_3 = 33.66 \cdot 10^{-6}$ °C^{-1}

Temperature difference
—$\Delta T = -280$ °C
Edges unconstrained

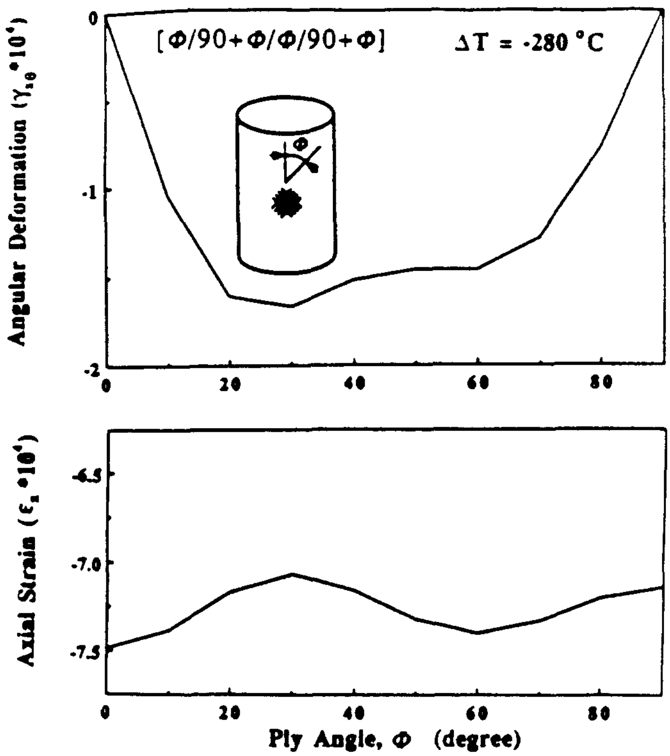

Fig. 6. The angular deformation ($\gamma_{x\theta} = dv/dx$) and the axial strain ($\varepsilon_x = du/dx$) of a $[\phi/90+\phi/\phi/90+\phi]$ cylinder cooled by $\Delta T = 280$ C (see Table 5).

there were only four plies in the cylinder. In a thicker cylinder the axial deformations may be affected more significantly by changes in ϕ. It is interesting to note that although the layup is balanced, the angular deformation is not zero. This means that the cylinder rotates about its axis except when $\phi = 0°$ and $\phi = 90°$.

The stress distributions in a [45/−45/45/−45] cylinder are presented in Fig. 7. In this figure only σ_x, σ_θ, σ_r, $\tau_{x\theta}$ are shown because the other two components τ_{xr}, $\tau_{\theta r}$ are zero. We observe that there are considerable shear stresses inside the cylinder. This is in contrast to the results of Hyer *et al.* in whose problem (cross-ply cylinder) the shear stresses are zero.

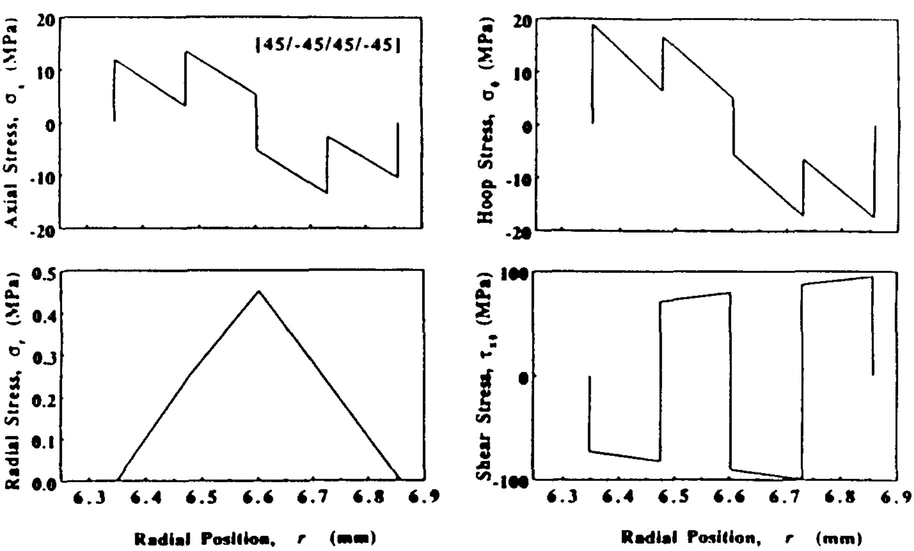

Fig. 7. The stress distribution in a [45/−45/45/−45] cylinder cooled by $\Delta T = 280$ C (see Table 5).

The aforementioned results illustrate that the ply orientation affects significantly the strains and stresses. Obviously, analyses developed for orthotropic cylinders should not be applied to cylinders with non-orthotropic ply orientations.

Acknowledgements—This work was supported by an Imre Korányi fellowship provided to LPK through the Thomas Cholnoky Foundation and by a National Defense Science and Engineering Grant to JMP.

REFERENCES

Hyer, M. W., Cooper, D. E. and Cohen, D. (1986). Stresses and deformations in cross-ply composite tubes subjected to uniform temperature change. *J. Thermal Stresses* **9**, 97–117.

Kollár, L. P. and Springer, G. S. (1992). Stress analysis of anisotropic laminated cylinders and cylindrical segments. *Int. J. Solids Structures* **29**, 1499–1517.

Korn, G. A. and Korn, T. M. (1968). *Mathematical Handbook for Scientists and Engineers* (2nd Edn). McGraw-Hill, New York.

Wolfram, S. (1988). *Mathematica. A System for Doing Mathematics by Computer.* Addison-Wesley, Redwood City.

APPENDIX A: THE SERIES OF THE TEMPERATURE LOAD

The given temperature $\Delta T(\theta, r)$ is expressed in the following series

$$\Delta T = \sum_{j=0}\left\{\left[\sum_{i=0}^{m} \Delta T_{ji}(r)^i\right]\cos j\theta + \left[\sum_{i=0}^{m} \Delta T^*_{ji}(r)^i\right]\sin j\theta\right\}. \tag{A1}$$

In the above expression ΔT in each ply is known. The objective is to find the constants ΔT_{ji} and ΔT^*_{ji} for each ply. At a given radius r_k ($k = 1, 2, \ldots, M$) the temperature can be expressed by the following Fourier series

$$\Delta T_k(\theta) = \sum_{j=0} \{\Delta \hat{T}_{jk} \cos j\theta + \Delta \hat{T}^*_{jk} \sin j\theta\} \tag{A2}$$

where the constants ΔT_{ji} and ΔT^*_{ji} are given by [see eqns (A5)–(A7)]

$$\Delta \hat{T}_{jk} = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \Delta T(\theta, r_k)\cos j\theta \, d\theta & \text{if} \quad j = 0 \\ \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \Delta T(\theta, r_k)\cos j\theta \, d\theta & \text{if} \quad j \geq 1 \end{cases} \tag{A3}$$

$$\Delta \hat{T}^*_{jk} = \begin{cases} 0 & \text{if} \quad j = 0 \\ \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \Delta T(\theta, r_k)\sin j\theta \, d\theta & \text{if} \quad j \geq 1. \end{cases} \tag{A4}$$

Equations (A3) and (A4) provide M values of the constants $\Delta \hat{T}_{jk}$, $\Delta \hat{T}^*_{jk}$ in each ply. Next, using the least square method (Korn and Korn, 1968) mth order polynomials ($m \leq M-1$) are fitted to the $\Delta \hat{T}_{jk}$ and the $\Delta \hat{T}^*_{jk}$ values. The coefficients of these polynomials are the constants ΔT_{j0}, $\Delta T_{j1}, \ldots, \Delta T_{j(m-1)}$, and ΔT^*_{j0}, $\Delta T^*_{j1}, \ldots, \Delta T^*_{j(m-1)}$.

The Fourier series of an arbitrary periodic function $f(\theta)$ with the period 2π is (Korn and Korn, 1968)

$$f(\theta) = \sum_{j=0} \{f_j \cos j\theta + f_j^* \sin j\theta\} \tag{A5}$$

where

$$f_j = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos j\theta \, d\theta & \text{if} \quad j = 0 \\ \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos j\theta \, d\theta & \text{if} \quad j \geq 1 \end{cases} \tag{A6}$$

$$f_j^* = \begin{cases} 0 & \text{if} \quad j = 0 \\ \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin j\theta \, d\theta & \text{if} \quad j \geq 1. \end{cases} \tag{A7}$$

APPENDIX B: EQUILIBRIUM CONDITIONS FOR THE LOADS

Substitution of the surface forces [eqn (17)] into the equilibrium conditions, eqns (2)–(5), results in the following equations

$$
\begin{aligned}
r^{o}\hat{p}^{o*}_{x0}-r^{i}\hat{p}^{i*}_{x0} &= 0\\
r^{o}\{\hat{p}^{o}_{r1}-\hat{p}^{o}_{\theta 1}\}-r^{i}\{\hat{p}^{i}_{r1}-\hat{p}^{i}_{\theta 1}\} &= 0\\
r^{o}\{\hat{p}^{o*}_{r1}+\hat{p}^{o*}_{\theta 1}\}-r^{i}\{\hat{p}^{i*}_{r1}+\hat{p}^{i*}_{\theta 1}\} &= 0\\
(r^{o})^{2}\hat{p}^{o*}_{\theta 0}-(r^{i})^{2}\hat{p}^{i*}_{\theta 0} &= 0.
\end{aligned}
\qquad (B1)
$$